Novel universal correlations in invariant random-matrix models

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We show that eigenvalue correlations in unitary-invariant ensembles of large random matrices adhere to novel universal laws that only depend on a multicriticality of the bulk density of states near the soft edge of the spectrum. Our consideration is based on the previously unknown observation that genuine density of states and $n-$point correlation function are completely determined by the Dyson's density analytically continued onto the whole real axis.

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Random matrices have been introduced in a physical context since the works by Wigner [1] and Dyson [2]. Initially proposed as an effective phenomenological model for description of the higher excitations in nuclei [3], the invariant ensembles of large random matrices found numerous applications in very diverse fields of physics such as two-dimensional quantum gravity [4], quantum chaos [5], and mesoscopic physics [6]. Apparently, this ubiquity owes its origin to the very idea of the construction of the invariant one-matrix model [7], which only reflects the fundamental symmetry (orthogonal, unitary or symplectic) of underlying physical system/phenomenon but discards its (irrelevant) microscopic details. Since the symmetry constraints follow from the first principles, even a rather crude matrix model allows identification of universal features which persist for a variety of systems with the same symmetry. This circumstance emphasizes the importance of the study of universality intrinsic in random matrices.

The simplest invariant random-matrix model is defined by the probability density

$$P[\mathbf{H}] = \frac{1}{\mathcal{Z}_N} \exp\{-\operatorname{Tr} V[\mathbf{H}]\} \tag{1}$$

of the entries H_{ij} of the $N \times N$ random matrix $\mathbf{H}$, where the function $V[\mathbf{H}]$ referred to as "confinement potential" must ensure existence of the partition function $\mathcal{Z}_N$, $N \gg 1$.

In the following we restrict our consideration to the unitary invariant, $U(N)$, matrix model. Nowadays it is widely believed that $U(N)$ invariant ensembles of large random matrices with rather strong level confinement may exhibit *three* different types of locally universal eigenlevel correlations which are characterized by the appropriately scaled two-point kernels.

• *Bulk scaling limit* is associated with a spectrum range where the confinement potential is well-behaved, and density of levels can approximately be taken as a constant. It has been proven in Refs. [8–10] that for rather strong confinement potentials [11] the two-point kernel follows the *universal sine law*

$$K_{\text{bulk}}(s, s') = \frac{\sin[\pi(s - s')]}{\pi(s - s')}. \tag{2}$$

Here scaling variable s is measured in the units of the mean level spacing: $s = \varepsilon/\Delta_N$.

• *Origin scaling limit* deals with that part of the spectrum where confinement potential displays logarithmic singularity: $V(\varepsilon) \to V(\varepsilon) - \alpha \log|\varepsilon|$. In the vicinity of the singularity $\varepsilon = 0$ the two-point kernel takes the *universal Bessel law* [12]:

$$K_{\text{orig}}(s, s') = \frac{\pi}{2}\sqrt{ss'}\,\frac{J_{\alpha+\frac{1}{2}}(\pi s)\, J_{\alpha-\frac{1}{2}}(\pi s') - J_{\alpha-\frac{1}{2}}(\pi s)\, J_{\alpha+\frac{1}{2}}(\pi s')}{s - s'}. \tag{3}$$

Here s is scaled by the level spacing near the origin, $s = \varepsilon/\Delta_N(0)$.

• *Soft-edge scaling limit*, relevant to the tail of eigenvalue support where crossover occurs from a non-zero density of states to a vanishing one [13], has been only investigated for Gaussian unitary ensemble (GUE) [14], and quite recently for $U(N)$ invariant ensembles of large random matrices associated with quartic and sextic confinement potentials [15]. It has been found that in the soft-edge scaling limit in all these ensembles the two-point kernels follow *Airy law*

$$K_{\text{soft}}(s, s') = \frac{\operatorname{Ai}(s)\operatorname{Ai}'(s') - \operatorname{Ai}(s')\operatorname{Ai}'(s)}{s - s'}. \tag{4}$$

Here $s \propto N^{2/3} \cdot (\varepsilon/D_N - 1)$ with D_N being the endpoint of the spectrum.

Whereas universality in the spectrum bulk and near its origin has rigorously been proven for a wide class of strong symmetric confinement potentials, the supposed universality of the Airy kernel has not been proven.

Our aim here is to demonstrate that the Airy correlations, Eq. (4), being universal for a wide class of matrix models Eq. (1), are indeed a particular case of more general novel universal correlations which are represented by the scaled $m-$th multicritical two-point kernel

$$K_{\rm soft}^{(m)}\left(s,s'\right)=\frac{{\rm G}\left(s|\nu^{*}\right){\rm G}'\left(s'|\nu^{*}\right)\cdot s^{\frac{3}{2}-\nu^{*}}-{\rm G}\left(s'|\nu^{*}\right){\rm G}'\left(s|\nu^{*}\right)\cdot\left(s'\right)^{\frac{3}{2}-\nu^{*}}}{s-s'},\tag{5}$$

where the function $\rm G$ is expressed through the Bessel functions as

$$\begin{aligned}{\rm G}\left(s|\nu^{*}\right)&=\frac{1}{2\sqrt{\nu^{*}}}\left[\sin\left(\frac{\pi}{4\nu^{*}}\right)+(-1)^{\nu^{*}-\frac{3}{2}}\right]^{-1/2}\\&\times\begin{cases}s^{\frac{1}{2}\left(\nu^{*}-\frac{1}{2}\right)}\left[I_{-\frac{1}{2}\left(1-\frac{1}{2\nu^{*}}\right)}\left(\frac{s^{\nu^{*}}}{\nu^{*}}\right)-I_{\frac{1}{2}\left(1-\frac{1}{2\nu^{*}}\right)}\left(\frac{s^{\nu^{*}}}{\nu^{*}}\right)\right], & s>0,\\|s|^{\frac{1}{2}\left(\nu^{*}-\frac{1}{2}\right)}\left[J_{-\frac{1}{2}\left(1-\frac{1}{2\nu^{*}}\right)}\left(\frac{|s|^{\nu^{*}}}{\nu^{*}}\right)+(-1)^{\nu^{*}-\frac{3}{2}}J_{\frac{1}{2}\left(1-\frac{1}{2\nu^{*}}\right)}\left(\frac{|s|^{\nu^{*}}}{\nu^{*}}\right)\right], & s<0,\end{cases}\end{aligned}\tag{6}$$

and parameter ν^{*} is determined by the even critical index [13] $m=0,2,4$, etc. of the matrix model:

$$\nu^{*}=m+\frac{3}{2}.\tag{7}$$

Note that the critical index m is completely determined by the type of singularity of the density of states near the soft edge [13]: $\nu_N\left(\varepsilon\right)=\left\langle{\rm Tr}\,\delta\left(\varepsilon-\mathbf{H}\right)\right\rangle\propto\left(1-\varepsilon^{2}/D_N^{2}\right)^{m+1/2}$.

Equations (5) - (7) together with Eqs. (25) and (26) below are the main results of the paper. Although we concentrate our attention on the problem of eigenvalue correlations near the soft edge, the treatment we present here is quite general being relevant to arbitrary spectrum range.

A. Within the orthogonal polynomial technique the two-point kernel $K_N\left(\varepsilon,\varepsilon'\right)$ determining the $n-$point correlation function R_n for eigenvalue spectrum of large random matrices, $R_n\left(\varepsilon_1,...,\varepsilon_n\right)=\det\left[K_N\left(\varepsilon_i,\varepsilon_j\right)\right]_{i,j=1...n}$, can be written through the fictitious "wavefunctions" $\psi_n\left(\varepsilon\right)$ as

$$K_N\left(\varepsilon,\varepsilon'\right)=c_N\frac{\psi_N\left(\varepsilon'\right)\psi_{N-1}\left(\varepsilon\right)-\psi_N\left(\varepsilon\right)\psi_{N-1}\left(\varepsilon'\right)}{\varepsilon'-\varepsilon}.\tag{8}$$

Here c_N is the recurrence coefficient entering three-term recurrence equation

$$\varepsilon P_{n-1}=c_nP_n+c_{n-1}P_{n-2},\;\;P_0\left(\varepsilon\right)=1,\;\;P_1\left(\varepsilon\right)=\varepsilon,\tag{9}$$

for polynomials P_n orthogonal on the whole real axis $\mathbf{R}$:

$$\int d\alpha\left(\varepsilon\right)P_n\left(\varepsilon\right)P_m\left(\varepsilon\right)=\delta_{nm},\tag{10}$$

and the "wavefunction" $\psi_n\left(\varepsilon\right)=P_n\left(\varepsilon\right)\exp\left\{-V\left(\varepsilon\right)\right\}$. The measure $d\alpha\left(\varepsilon\right)=\exp\left\{-2V\left(\varepsilon\right)\right\}d\varepsilon$ is completely determined by symmetric confinement potential

$$V\left(\varepsilon\right)=\sum_{k=1}^{p}\frac{d_k}{2k}\varepsilon^{2k}\tag{11}$$

with $d_p>0$. The signs of the rest d_k's can be arbitrary but they should lead to an eigenvalue density supported on a single connected interval $\left(-D_N,+D_N\right)$.

To study the eigenvalue correlations in the random matrix ensemble with confinement potential Eq. (11) we note that a three-term recurrence equation for orthogonal polynomials $P_n\left(\varepsilon\right)$ can be mapped onto a second order differential equation for these orthogonal polynomials and/or corresponding wavefunctions $\psi_n\left(\varepsilon\right)$. This was already observed for the first time by J. Shohat in 1930 [16]. Considerably later Shohat's idea was developed by Bonan and Clark [17]. The simple and elegant method proposed in Refs. [16,17] turns out to be a very general and powerful one for analysis of spectral properties possessed by large random matrices.

To map Eq. (9) onto a second order differential equation for ψ_n, we note that the following identity takes place:

$$\frac{dP_n}{d\varepsilon}=A_n\left(\varepsilon\right)P_{n-1}-B_n\left(\varepsilon\right)P_n,\tag{12}$$

where the functions $A_n\left(\varepsilon\right)$ and $B_n\left(\varepsilon\right)$ can be found from consideration below. Since $dP_n/d\varepsilon$ is a polynomial of the degree $n-1$ it can be represented [18] through the Fourier expansion in terms of the kernel $K_n\left(t,\varepsilon\right)=\sum_{k=0}^{n-1}P_k\left(t\right)P_k\left(\varepsilon\right)$ as follows:

$$\frac{dP_n}{d\varepsilon}=\int d\alpha\left(t\right)\frac{dP_n}{dt}K_n\left(t,\varepsilon\right).\tag{13}$$

Integrating by part we obtain that

$$\frac{dP_n}{d\varepsilon}=2\int d\alpha\left(t\right)K_n\left(t,\varepsilon\right)\left(\frac{dV}{dt}-\frac{dV}{d\varepsilon}\right)P_n\left(t\right).\tag{14}$$

Now, making use of the Christoffel-Darboux theorem [18] we conclude that unknown functions A_n and B_n in Eq. (12) are

$$A_n\left(\varepsilon\right)=2c_n\int d\alpha\left(t\right)\frac{V'\left(t\right)-V'\left(\varepsilon\right)}{t-\varepsilon}P_n^{2}\left(t\right),\tag{15}$$

$$B_n\left(\varepsilon\right)=2c_n\int d\alpha\left(t\right)\frac{V'\left(t\right)-V'\left(\varepsilon\right)}{t-\varepsilon}P_n\left(t\right)P_{n-1}\left(t\right).\tag{16}$$

At first glance representations Eqs. (15) and (16) are rather useless as far as they involve the same orthogonal

polynomials which enter Eq. (12). Nevertheless these expressions do allow us to get the functions A_n and B_n in closed forms directly related to confinement potential and to endpoint of eigenvalue spectrum. Restricting our following consideration to large indices $n = N \gg 1$ we reduce Eq. (9) to the asymptotic form

$$\varepsilon P_N = c_N \left(P_{N+1} + P_{N-1}\right), \tag{17}$$

where

$$\varepsilon^\lambda P_N = \sum_{j=0}^{\lambda} \left(\begin{array}{c} \lambda \\ j \end{array} \right) c_N^\lambda P_{N+2j-\lambda},\ \lambda \geq 0. \tag{18}$$

Substituting $V(\varepsilon)$ given by Eq. (11) into Eq. (15) yields

$$A_N(\varepsilon) = 2c_N \sum_{k=1}^{p} \sum_{\lambda=1}^{2k-1} d_k \varepsilon^{\lambda-1} \int d\alpha(t)\, P_N^2(t)\, t^{2k-\lambda-1}. \tag{19}$$

Then, taking into account Eq. (18) as well as the orthogonality of P_n we arrive at the expression defined for *arbitrary* ε:

$$A_N(\varepsilon) = 2c_N \sum_{k=1}^{p} d_k \sum_{\lambda=1}^{k} \left(\begin{array}{c} 2(k-\lambda) \\ k-\lambda \end{array} \right) c_N^{2k-\lambda} \varepsilon^{2\lambda-2}. \tag{20}$$

It is easy to verify that $A_N(\varepsilon)$ can be represented through the *Dyson's density*

$$\nu_D(\varepsilon) = \frac{2}{\pi^2} \mathcal{P} \int_0^{D_N} \frac{t\,dt}{t^2 - \varepsilon^2} \frac{dV}{dt} \sqrt{\frac{1-\varepsilon^2/D_N^2}{1-t^2/D_N^2}} \tag{21}$$

defined on the whole real axis $\mathbf{R}$:

$$A_N(\varepsilon) = \frac{\pi \nu_D(\varepsilon)}{\sqrt{1-\varepsilon^2/D_N^2}}. \tag{22}$$

Here ε takes *arbitrary* value that can lie both inside and outside of an eigenvalue support. The spectrum endpoint $D_N = 2c_N$ is the positive root of the integral equation

$$N = \frac{2}{\pi} \int_0^{D_N} \frac{dV}{dt} \frac{t\,dt}{\sqrt{D_N^2 - t^2}} \tag{23}$$

following from normalization of Dyson's density.

Combining Eqs. (12), (17), (22), and using asymptotic identity

$$B_N = \frac{\varepsilon}{D_N} A_N - \frac{dV}{d\varepsilon}, \tag{24}$$

which is a consequence of Eqs. (15) and (16), it is a straightforward step to reach the following remarkable asymptotic differential equation:

$$\psi_N'' - \left[\frac{d}{d\varepsilon} \log \left(\frac{\pi \nu_D(\varepsilon)}{\sqrt{1-\varepsilon^2/D_N^2}} \right) \right] \psi_N' + \pi^2 \nu_D^2(\varepsilon)\, \psi_N = 0, \tag{25}$$

which together with relationship

$$\psi_N' = \frac{\pi \nu_D(\varepsilon)}{\sqrt{1-\varepsilon^2/D_N^2}} \left(\psi_{N-1} - \frac{\varepsilon}{D_N} \psi_N \right) \tag{26}$$

provide a general basis for the study of eigenvalue correlations in *arbitrary spectral range* [19].

An interesting property of these equations is that they do not contain confinement potential explicitly, but only involve the *Dyson's density* ν_D and spectrum endpoint D_N. Moreover, it turns out that the knowledge of Dyson's density (that coincides with *real* density of states only in the spectrum bulk) is sufficient to determine the *genuine* density of states, as well as the $n-$point correlation function, *everywhere*. We also note that Eq. (25) can be derived in a different way for monotonous confinement potentials increasing at least as fast as $|\varepsilon|$ at infinity. This suggests that differential equation Eq. (25) should hold generally and not only for confinement potentials having polynomial form Eq. (11).

B. Up to this point our derivation was quite general without any respect to the soft edge of eigenvalue support. We now focus our attention on the eigenvalue correlations near the soft edge $\varepsilon = D_N$. It is known [13] that tuning coefficients d_k which enter V one can reach a situation when the bulk (Dyson's) density of states will possess a singularity of the type:

$$\nu_D(\varepsilon) = \left[1 - \frac{\varepsilon^2}{D_N^2} \right]^{m+1/2} \mathcal{R}_N \left(\frac{\varepsilon}{D_N} \right) \tag{27}$$

with $m = 0, 2, 4$, etc., and $\mathcal{R}_N$ being a well-behaved function with $\mathcal{R}_N(1) \neq 0$. [Odd indices m are inconsistent with our choice for leading coefficient d_p, entering confinement potential $V(\varepsilon)$, be positive in order to keep a convergence of integral for partition function $\mathcal{Z}_N$ in Eq. (1)]. We intend to demonstrate that as long as multicriticality of order m is reached, the eigenvalue correlations in the vicinity of the soft edge become universal, and are independent of the particular potential chosen. The order m of the multicriticality is the only parameter which governs spectral correlations in the soft-edge scaling limit.

Let us move the spectrum origin to its endpoint D_N, making replacement

$$\varepsilon_s = D_N \left[1 + s \cdot \frac{1}{2} \left(\frac{2}{\pi D_N \mathcal{R}_N(1)} \right)^{1/\nu^*} \right], \tag{28}$$

that defines the $m-$th *soft-edge scaling limit* provided $s \ll \left(D_N \mathcal{R}_N(1)\right)^{1/\nu^*} \propto N^{1/\nu^*}$. It is straightforward to show from Eqs. (25) and (26) that the function $\widehat{\psi}_N(s) = \psi_N(\varepsilon_s - D_N)$ obeys differential equation

$$\widehat{\psi}_N''(s) - \frac{\left(\nu^* - \frac{3}{2}\right)}{s}\widehat{\psi}_N'(s) - s^{2(\nu^*-1)}\widehat{\psi}_N(s) = 0, \quad (29)$$

and that the following relation takes place:

$$\widehat{\psi}_{N-1}(s) = \widehat{\psi}_N(s) + (-1)^{\nu^* - \frac{3}{2}} \left(\frac{2}{\pi D_N \mathcal{R}_N(1)}\right)^{\frac{1}{2\nu^*}} s^{\frac{3}{2}-\nu^*}\widehat{\psi}_N'(s). \quad (30)$$

Solution to Eq. (29) which decreases at $s \to +\infty$ (that is at far tails of the density of states) is given (up to an arbitrary factor λ_N) by the function $\mathrm{G}(s|\nu^*)$, Eq. (6). The factor λ_N can be found by fitting [15] the density of states $K_N(\varepsilon_s, \varepsilon_s)$, Eq. (8), to the bulk density of states, Eq. (27), near the soft edge provided $1 \ll s \ll N^{1/\nu^*}$. Then, making use of Eqs. (6), (8) and (30) we easily obtain that in the $m-$th soft-edge scaling limit, Eq. (28), the two-point kernel

$$K_{\mathrm{soft}}^{(m)}(s, s') = \lim_{N\to\infty} K(\varepsilon_s, \varepsilon_{s'})\frac{d\varepsilon_s}{ds} \quad (31)$$

is determined by Eq. (5). In particular case of $m = 0$, that is inherent in random-matrix ensembles with monotonous confinement potential, the function G coincides with Airy function, $\mathrm{G}\left(s|\frac{3}{2}\right) = \mathrm{Ai}(s)$, and the Airy correlations, Eq. (4), are recovered.

It follows from Eqs. (5) and (29) that density of states in the same scaling limit

$$\nu_{\mathrm{soft}}^{(m)}(s) = \left(\frac{d}{ds}\mathrm{G}(s|\nu^*)\right)^2 s^{\frac{3}{2}-\nu^*} - \left[\mathrm{G}(s|\nu^*)\right]^2 s^{\nu^*-\frac{1}{2}} \quad (32)$$

is also universal.

The large-$|s|$ behavior of $\nu_{\mathrm{soft}}^{(m)}$ can be deduced from the known asymptotic expansions of the Bessel functions:

$$\nu_{\mathrm{soft}}^{(m)}(s) = \begin{cases} \frac{|s|^{\nu^*-1}}{\pi} + \frac{(-1)^{\nu^*-\frac{1}{2}}}{4\pi|s|}\cos\left(\frac{2|s|^{\nu^*}}{\nu^*}\right), & s \to -\infty, \\ \frac{\exp\left(-\frac{2s^{\nu^*}}{\nu^*}\right)}{4\pi s}\frac{\cos^2\left(\frac{\pi}{4\nu^*}\right)}{\sin\left(\frac{\pi}{4\nu^*}\right) + (-1)^{\nu^*-\frac{3}{2}}}, & s \to +\infty. \end{cases} \quad (33)$$

Note that the leading order behavior as $s \to -\infty$ is consistent with the $|s|^{\nu^*-1}$ singularity of the bulk density of states, Eq. (27).

To conclude, in this paper we presented a general formalism for a treatment of eigenlevel correlations in spectra of $U(N)$ invariant ensembles of large random matrices with strong level confinement. An important ingredient of our analysis is the second order differential equation which connects the Dyson's density with a fictitious "wavefunction" ψ_N which is needed for calculations of eigenvalue correlations within the framework of orthogonal polynomial technique. This consideration is relevant to arbitrary energy range. We have applied this formalism to examine the eigenlevel correlations near the endpoint of single spectrum support. It has been shown that in the soft-edge scaling limit there are novel universal eigenlevel correlations which only depend on the even multicritical index of a matrix model. In a particular case $m = 0$, corresponding to monotonous confinement potentials, universality of the Airy correlations is recovered.

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